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Passive scalar convection in a 2D long-range delta-correlated velocity field: exact results

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Abstract. The paper presents a new field-theoretical approach to a 2D passive scalar advected by long-range random delta-correlated-in-time velocity field. The Gaussian form of the distribution for the stretching rate of a passive scalar cloud is derived and its parameters are found explicitly.

1. Definition of the model

The convection of a passive scalar (PS) in two dimensions by an external long-range velocity field is a well defined linear problem [1, 2]. To solve the problem is to find all the correlation functions of the passive scalar field $w(r;t)$ governed by the following equation:

$$\dot{\omega} + u_\alpha \nabla_\alpha \omega = \phi \tag{1}$$

where $u(r;t)$ is an external Eulerian long-range velocity field (the spatial Fourier-harmonic of the velocity field $u_k(t)$ is non-zero only at $k < L^{-1}$) and ϕ is a random external source localized in momentum space at $k_0 = L^{-1}$. The measure of averaging over the random source in the simplest case of white-noise statistics of the source can be chosen in the following form without any loss of generality as shown in [3]

$$\mathcal{D}\phi \exp\left(-\int_{-\infty}^{+\infty} \phi^2_{k=1/L}(t)\,\mathrm{d}t/2P_2 \right)$$

$$\phi_k = (2\pi)^{-2} \int \phi(r;t) \exp(\mathrm{i}k\cdot r)\,\mathrm{d}r$$

$$\langle \phi_k(t)\phi_{k'}(t')\rangle_s \equiv \Xi_{k-k'}(t-t') = \frac{P_2}{2\pi k_0}\delta(t-t')\delta(k-k')\delta(k-k_0). \tag{2}$$

Here P_2 stands for the flow of the squared PS on the source.

In order to eliminate a homogeneous sweeping, it is convenient to pass to the locally co-moving frame [4] expressing equation (1) in terms of the quasi-Lagrangian (QL) velocities related to the initial Eulerian velocities by means of the formula

$$u(r;t) = v\Big(r - \int^t v(0;\tau)\,\mathrm{d}\tau; t \Big). \tag{3}$$

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Correspondingly, equation (1) takes the form

$$\dot{\omega} + (v^{\alpha} - v_0^{\alpha})\nabla_{\alpha}\omega = \phi \qquad v_0 = v(0;\tau). \tag{4}$$

Following the ideas of a recent paper [3], one can expand $v^{\alpha}(r) - v^{\alpha}(0) = \sigma^{\alpha\beta}r^{\beta}$ for the points satisfying $r < L$. Here, $\hat{\sigma}$ is the random-in-time traceless-symmetric†‡ matrix of velocity derivatives. For the *white-noise velocity-field statistics* (which is the only case considered in the present paper), the measure of the $\hat{\sigma}$ elements has the form providing space-isotropy of the simultaneous-pair correlator of the QL velocities

$$\mathcal{D}\hat{\sigma}(t) = \mathcal{D}\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \mathcal{D}a\mathcal{D}b\exp(-S_0) \qquad S_0 = \frac{1}{2D}\int (a^2+b^2)\,\mathrm{d}t. \tag{5}$$

To solve the resulting equation

$$\dot{\omega} + \sigma^{\alpha\beta}r^{\beta}\nabla_{\alpha}\omega = \phi \tag{6}$$

one performs the substitutions

$$\omega(r;t) = \psi(R;t) \qquad R^{\alpha} = W^{\alpha\beta}(t)r^{\beta} \tag{7}$$

that allows the solution of equation (6) to be written in the following form:

$$\omega(r;t) = \psi(R(t);t) = \int_{-\infty}^{t}\mathrm{d}\tau\,\phi(\hat{W}^{-1}(\tau,-\infty)R(t);\tau) \equiv \int_{-\infty}^{t}\mathrm{d}\tau\,\phi(\hat{W}(t,\tau)r;\tau) \tag{8}$$

where the matrix $\hat{W}(t;\tau)$ can sought as a solution to the relation

$$\dot{\hat{W}}_t + \hat{W}\hat{\sigma} = 0. \tag{9}$$

The resulting equation (9) describes the advection of a unit cell (a cloud of the PS) embedded in an external large-scale velocity field. The splitting of the velocity derivatives matrix (6) into diagonal and off-diagonal parts results in a corresponding pure stretching and a pure shearing of the cell. Volume conservation is guaranteed by the zero trace of the matrix. The dropped antisymmetric part of the $\hat{\sigma}$ matrix‡ describes a unit-cell rotation. Using equation (8) we can rewrite the correlation functions of the PS field in terms of the correlation functions of the source field ϕ. For example, the pair-correlation function is

$$\langle\omega(r_1;t_1)\omega(r_2;t_2)\rangle = \int_{-\infty}^{t_1}\mathrm{d}\tau_1\int_{-\infty}^{t_2}\mathrm{d}\tau_2\langle\Xi(\hat{W}(t_1,\tau_1)r_1 - \hat{W}(t_2,\tau_2)r_2;\tau_1-\tau_2)\rangle \tag{10}$$

where $\langle\ldots\rangle$ stands for the average with respect to the measure (5). The explicit form of the simultaneous-pair correlator (which has to be isotropic) can be extracted from equations (10) and (2) as

$$\langle\omega(r_1;t)\omega(r_2;t)\rangle = P_2\int_0^{\infty}\mathrm{d}t\left\langle J_0\left(\frac{|\hat{W}(t,0)(r_1-r_2)|}{L}\right)\right\rangle \tag{11}$$

† Due to the fact that one assumes that the flow is inviscid.

‡ It is possible to show that in the case of the white-noise statistics the arbitrary asymmetry (which is compatible with the isotropy of the velocity field) does not affect any physical averages at all (see also [3]).

with $J_0(x)$ being the Bessel function of zeroth order.

So, following the calculations in [3], we have reduced our initial problem to the evaluation of some functions of matrix $\hat{W}$ satisfying equation (9). The statistics of the matrix elements of $\hat{W}(t, 0)$ is defined by the ensemble of random real traceless symmetric matrices $\hat{\sigma}$, see equation (5). The authors of [3] have calculated $\langle|\hat{W}(t, 0)r|\rangle$ by means of the direct expansion of the anti-chronological T-exponent that is the formal solution to equation (9). This enabled them to estimate the simultaneous-pair correlator (11) at small distances $r_1 - r_2$ as $(P_2/D)\ln(L/r)$. In addition, by analysing the set of many-point correlation functions of the PS field, Falkovich and Lebedev [3] were able to demonstrate that the instantaneous statistics of such a field becomes closer to Gaussian statistics when passing downscales.

In the present paper, we suggest a new non-perturbative field-theoretical approach to calculating the PS-field correlations. We show that the asymptotic behaviour of the pair-correlator equation (11) is governed by the self-averaging (or 'deterministic') quantity: the Lyapunov exponent. By means of our method we rederive, extend and exactly prove all the above-mentioned results by Falkovich and Lebedev [3]. In particular, we explicitly find the value of the variance of the Gaussian distribution characterizing the short-range PS-field statistics.

2. Functional representation for averaged functionals of $\hat{W}$

The matrix $\hat{W}(t)$ can be extracted from equation (9) only in the form of the anti-chronological time-ordered exponent

$$\hat{W}(t, 0) = \tilde{T} \exp\left(-\int_0^t \mathrm{d}t'\, \hat{\sigma}(t')\right) \qquad \hat{W}(0, 0) = \hat{1}. \tag{12}$$

rather than in terms of some regular function of $\hat{\sigma}$ that reflects the strong interaction between the pure stretching and shearing processes. A similar problem—transformation of a time-ordered exponent of some linear combination of spin $SU(2)$ operators (arising when one tries to write down an exact functional representation for the partition function of the quantum Heisenberg ferromagnet)—has been solved [5, 6]. The main objective of the method is to introduce a new set of integration variables in the functional integral such that $\tilde{T}\exp$ becomes a regular function when expressed in these terms. In the present context we use a new modification of the ansatz proposed in [6], expanding the σ matrix over the spin 2×2 matrices as

$$\hat{\sigma} = a\hat{\sigma}_z + b\hat{\sigma}_x. \tag{13}$$

First, we introduce a new basis of the spin algebra

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \hat{\sigma}_+ = \hat{\sigma}_z + \mathrm{i}\hat{\sigma}_x = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix} \qquad \hat{\sigma}_- = \hat{\sigma}_z - \mathrm{i}\hat{\sigma}_x = \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} \tag{14}$$

that corresponds to the rotation of the quantization axis from the usual orientation (parallel to the z-axis) to a new orientation parallel to the y-axis. Instead of the fields $a(t), b(t)$, we choose new ones $\varphi^{\pm} = (a \pm \mathrm{i}b)/2$ which transform $\hat{\sigma}$ and the integration measure equation (5) to the more compact form

$$\hat{\sigma} = \varphi^{-}\hat{\sigma}_{+} + \varphi^{+}\hat{\sigma}_{-} \qquad \mathcal{D}\hat{\sigma}(t) = \mathcal{D}\varphi^{\pm}\exp(-S_0) \qquad S_0 = \frac{2}{D}\int_0^{+\infty} \varphi^{+}\varphi^{-}\,\mathrm{d}t. \tag{15}$$

Let us now introduce the operator given in explicit form as

$$\hat{A}(t,0) = \exp[-\hat{\sigma}_{-}\psi^{+}(0)]\exp\left[-\hat{\sigma}_{+}\int_0^t \psi^{-}(t')\exp\left(2\int_0^{t'}\rho(t'')\,\mathrm{d}t''\right)\mathrm{d}t'\right] \\ \times\exp\left[\hat{\sigma}_y\int_0^t \rho(t')\,\mathrm{d}t'\right]\exp[\hat{\sigma}_{-}\psi^{+}(t)] \tag{16}$$

where $\psi^{\pm}$ and ρ are some new dynamical fields. Using the commutation relations for the spin operators $\sigma_{\pm}, \sigma_y$, it is easy to check that the operator $\hat{A}$ obeys the differential equation

$$\dot{\hat{A}}_t = \hat{A}[-\hat{\sigma}_{+}\psi^{-} + \hat{\sigma}_{-}(4\psi^{-}(\psi^{+})^2 - 2\rho\psi^{+} + \dot{\psi}^{+}) + \hat{\sigma}_y(-4\psi^{-}\psi^{+} + \rho)]. \tag{17}$$

Using the fact that the first exponential factor in equation (16) makes the operator $\hat{A}$ satisfy the condition $\hat{A}(0,0) = 1$ and comparing the equation (17) with the equivalent one (9), we find the change of variables

$$\varphi^{-} = \psi^{-} \qquad \varphi^{+} = -\dot{\psi}^{+} + 4\psi^{-}(\psi^{+})^2 \tag{18}$$

$$\rho = 4\psi^{-}\psi^{+} \tag{19}$$

for the case in which dynamical matrices $\hat{W}$ and $\hat{A}$ coincide. This allows us to obtain the explicit functional integral representation for any average written in terms of $\hat{W}$ by changing variables from $\varphi^{\pm}$ to $\psi^{\pm}$. The Jacobian of map equation (18) is

$$\mathcal{D}\varphi^{\pm} = \mathcal{J}[\psi^{\pm}]\mathcal{D}\psi^{\pm} \tag{20}$$

and it is essentially dependent on the type of regularization and conditions imposed on the field ψ^{+}. The problem is that the transformation equation (18) contains the derivative of the field ψ^{+} with respect to time in its right-hand side. Therefore, it should be supplied with some initial or boundary conditions. In [6], it has been shown that only by imposing initial conditions is it possible to ensure the invertability of the map equation (18). We also note that in the course of the calculation it is necessary to average some functions of the operator $A(T,0)$ at fixed time moment T over the measure equation (15). For a given T it is convenient to fix the final value of the field ψ^{+}

$$\psi^{+}(T) = -\tfrac{1}{2}. \tag{21}$$

Then the operator $A(T,0)$ acting on the initial vector $\binom{1}{0}$ produces the following simple expressions

$$R(T) = \hat{A}(T,0)\begin{pmatrix}1\\0\end{pmatrix} = \tfrac{1}{2}\exp\left(4\int_0^T \psi^{+}\psi^{-}\,\mathrm{d}t\right)\begin{pmatrix}1-2\psi^{+}(0)\\ \mathrm{i}(1+2\psi^{+}(0))\end{pmatrix} \tag{22}$$

$$R^2(T) = -2\psi^{+}(0)\exp\left(8\int_0^T \psi^{+}\psi^{-}\,\mathrm{d}t\right). \tag{23}$$

Here we have exploited the isotropy condition and substituted $R(0) = \binom{1}{0}$ for the initial value of the vector $R(T)$ without any loss of generality.

The regularization of the map equation (18) [6,7] is determined by physical argumentation. Here it stems from the translation-in-time invariance of the white-noise measure equation (15). Indeed, the equality

$$\left\langle \varphi^+(T) \int^T \varphi^-(t)\,\mathrm{d}t \right\rangle = \left\langle \varphi^-(T) \int^T \varphi^+(t)\,\mathrm{d}t \right\rangle \tag{24}$$

leads to the extension of the definition of the step function $\theta(x)$ at $x = 0$ such that $\theta(0) = \frac{1}{2}$. Thus, the discrete version of the map (15) ($\varphi_n^\pm = \varphi^\pm(t_n)$; $n = 1, \ldots, M$; $\epsilon = T/M \to 0$; $t_n = \epsilon n$; $M \to \infty$), compatible with the symmetry condition equation (24), is

$$\varphi_n^- = \psi_n^- \qquad \varphi_n^+ = -\frac{1}{\epsilon}(\psi_{n+1}^+ - \psi_n^+) + \psi_n^-(\psi_n^+ + \psi_{n+1}^+)^2 \tag{25}$$

which gives the following expression for the Jacobian:

$$\mathcal{J} = \text{constant} \times \exp\left(4 \int_0^T \psi^+\psi^-\,\mathrm{d}t'\right). \tag{26}$$

When calculating the Jacobian equation (26), the fields $\varphi^\pm$ and $\psi^\pm$ were considered to be independent complex variables or, in other words, as different coordinate systems in the whole space C^{2M} of the fields' configurations. As the conditions

$$\varphi^+ = (\varphi^-)^* \tag{27}$$

are externally imposed on the model they specify the surface Σ in C^{2M} along which the differential forms $\mathcal{D}\varphi^+ \bigwedge \mathcal{D}\varphi^-$ or $\mathcal{D}\psi^+ \bigwedge \mathcal{D}\psi^-$ are integrated. For the coordinate sets $(\psi^\pm)$, equation (18) for Σ can be considered to be implicit. According to the Cauchy–Poincaré theorem, the integration surface can be deformed in an arbitrary way in the convergence domain provided we integrate an analytical function. There exists a continuous family of surfaces (homotopy) which includes both Σ and the 'standard' Σ':

$$\Sigma' = \left\{-\psi^+ = (\psi^-)^*\right\}. \tag{28}$$

The explicit expression for such a homotopy in a more general case can be found in [7]. Thus, we can replace the surface of integration Σ by the standard Σ'.

Substituting equation (18) into the measure equation (5) and using expressions (20) and (26), we obtain the following modification of the measure for averaging over $\psi^\pm$†:

$$N\mathcal{D}\psi^\pm \exp(-S_1\{\psi^\pm\}) \qquad S_1 = \int_0^{+\infty} \left[-\frac{2}{D}\dot{\psi}^+\psi^- + \frac{8}{D}(\psi^-\psi^+)^2 - 4\psi^+\psi^-\right] \mathrm{d}t \tag{29}$$

or, correspondingly, in the discrete form

$$S_1 = \sum_n \left[-\frac{2}{D}(\psi_{n+1}^+ - \psi_n^+)\psi_n^- + \frac{2}{D}\epsilon(\psi_n^-)^2(\psi_n^+ + \psi_{n+1}^+)^2 - 2\epsilon\psi_n^-(\psi_n^+ + \psi_{n+1}^+)\right]. \tag{30}$$

This means that we have reformulated the initial problem of the multiplicative random-*matrix* process with the measure equation (5) to the multiplicative random-*scalar* process with the measure given by equation (29).

† It should be noted that the convergence is provided by the term $-\dot{\psi}^+\psi^-$ in the action where the discretization (25) is assumed.

3. Gaussianity of passive scalar correlations

Another peculiarity of the arbitrary multiplicative random process is the Gaussian-like fluctuations of the exponential rate $\lambda(T)$ with an amplitude that decays like constant $/\sqrt{T}$ when T goes to infinity. Let us note that this fact has been proven both for the multiplicative random scalar process (see, e.g. [8]) and matrix process (see [9]). The usual way to calculate the constant in front of $1/\sqrt{T}$ is to restore the distribution function of $\lambda(t)$ at arbitrary time from the set of moments $\langle R^{2n}\rangle$ and then extract the Gaussian-like peak around the mean Lyapunov exponent at large enough time (for an example of such a calculation in 1D see [10] and for quasi 1D see [11] localization problems). We follow a similar procedure in the present context.

It is convenient to define the exponential stretching rate in our case as

$$\lambda \equiv \ln[R^2(T)]/2T. \tag{31}$$

In order to find the probability distribution function (PDF) of λ let us first calculate all the even moments of R

$$R_m(T) \equiv \langle R^{2m}\rangle = \frac{\left\langle(-2\psi^+(0))^m \exp\left(8m\int_0^T \psi^+\psi^- \,\mathrm{d}t\right)\right\rangle_1}{\langle 1\rangle_1} \tag{32}$$

where $\langle\ldots\rangle_1$ stands for the average with respect to the measure equation (29). Passing to the new integration variables by means of the gauge transformations

$$\psi^\pm = \chi^\pm \exp\left[\mp 4\int_t^T \chi^+\chi^- \,\mathrm{d}t' \pm (2m+1)D(T-t)\right] \tag{33}$$

which can be written in discretized form as

$$\psi_n^+ = \chi_n^+ \exp\left[-2\sum_{j=n}^{M}\epsilon(\chi_{j+1}^+ + \chi_j^+)\chi_j^- + (2m+1)\epsilon D(M-n+1)\right]$$

$$\begin{aligned}\psi_n^- = \chi_n^- \exp\Bigg[&2\sum_{j=n+1}^{M}\epsilon(\chi_{j+1}^+ + \chi_j^+)\chi_j^- + \epsilon(\chi_n^+ + \chi_{n+1}^+)\chi_n^- \\ &- (2m+1)\epsilon D(M-n) - \tfrac{1}{2}\epsilon(2m+1)D\Bigg]\end{aligned} \tag{34}$$

we reduce the averaging in both numerator and denominator (the latter case corresponds to $m=0$) of equation (32) to the Gaussian type

$$\begin{aligned}&\left\langle(-2\psi^+(0))^m \exp\left(8m\int_0^T \psi^+\psi^- \,\mathrm{d}t\right)\right\rangle_1 = (-2)^m\langle(\chi^+(0))^m\rangle_2 \exp\left\{\left[(2m+2)m+\tfrac{1}{2}\right]DT\right\}\\ &\langle 1\rangle_1 = \mathrm{e}^{DT/2}.\end{aligned} \tag{35}$$

Here $\langle\ldots\rangle_2$ stands for the averaging with respect to the Gaussian measure

$$N'\mathcal{D}\chi^\pm \exp(-S_2\{\chi^\pm\}) \qquad S_2 = \frac{2}{D}\int_0^{+\infty} \dot{\chi}^+\chi^- \,\mathrm{d}t \tag{36}$$

and one takes into account the fact that the Jacobian of the map equation (33) is

$$J[\chi] = \exp\left[-2\int_0^T \chi^+\chi^- + (m+\tfrac{1}{2})D(T-t)\right]. \tag{37}$$

The average $\langle(\chi^+(0))^m\rangle_2$ is equal to $(-\frac{1}{2})^m$ due to the condition $\chi^+(T) = -\frac{1}{2}$. This result is easy to get by shifting $\chi^+ \to -\frac{1}{2} + \tilde{\chi}^+$ and noticing that all the averages of $\tilde{\chi}^+$ are equal to zero. Thus, we arrive at the result

$$R_m(T) = \exp(DT(2m+2)m). \tag{38}$$

By knowing R_m one can extract the Fourier representation for the PDF of R^2 in the following way:

$$\tilde{\mathcal{P}}_k = \sum_{m=0}^{\infty} \frac{(ik)^m}{m!} R_m = \int_{-\infty}^{+\infty} \frac{e^{-DT/2}}{\sqrt{8\pi DT}} \exp\left(-\frac{x^2}{8DT} + \frac{x}{2} + ike^x\right) \tag{39}$$

and the PDF for R^2 is obtained after Fourier inversion of the form

$$\tilde{\mathcal{P}}(R^2) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ikz}\tilde{\mathcal{P}}_k = \int_{-\infty}^{+\infty} dx\, \frac{e^{-DT/2}}{\sqrt{8\pi DT}} \exp\left[-\frac{x^2}{8DT} + \frac{x}{2}\right]\delta(R^2 - e^x). \tag{40}$$

Correspondingly, the PDF of the stretching rate defined in equation (30) is explicitly given by the expression

$$\mathcal{P}(\lambda) = \sqrt{\frac{T}{2\pi D}} \exp\left(-\frac{T(\lambda - D)^2}{2D}\right) \tag{41}$$

which shows *exact Gaussian fluctuations* around the average value $\langle\lambda\rangle = D$ with the variance D/T vanishing when T tends to infinity. In particular, λ tends asymptotically to a non-random ('deterministic') quantity (the Lyapunov exponent) in agreement with the general theory.

In order to pass from the statistics of λ to the statistics of the PS-field, we introduce the fluctuating quantity

$$Q\left(\frac{r}{L}\right) \equiv w(r_1;t)w(r_2;t) = P_2 \int_0^{\infty} dt\, J_0\left(\frac{r}{L}e^{\lambda t}\right) \tag{42}$$

the distribution of which can be restored from the PDF of λ. Indeed, at $r/L \ll 1$ we can cut the integral in the right-hand side of equation (42) at $t = \ln(r/L)/\lambda$ and estimate Q with logarithmic accuracy as

$$Q\left(\frac{r}{L}\right) \approx \frac{P_2}{\lambda(\ln(L/r)/\lambda)} \ln\left(\frac{L}{r}\right). \tag{43}$$

This gives us the possibility of extracting the PDF of Q from expression (41). We obtain

$$\mathcal{P}(Q) \approx \frac{D}{P_2\sqrt{2\pi \ln(L/r)}} \exp\left[-\frac{(Q - (P_2/D)\ln(L/r))^2}{2P_2 Q} D\right]. \tag{44}$$

Such a PDF has, at $r \to 0$ asymptotically, the form of a Gaussian distribution for the quantity $Q/\ln(L/r)$ approaching the δ-functional form when $L/r \to \infty$.

4. Conclusion

We have described explicitly the local (no space averaging) statistics of a passive scalar advected by a long-range delta-correlated velocity field. First, by following [3], we transformed the initial problem to a study of the advection of a unit cell embedded in a long-range velocity field. The second key step in the evaluation of the problem was an averaging of the time-ordered exponent describing the cell advection. We applied a modification of the functional-integral technique, used before in the theory of magnetism [6], to obtain the statistics of the stretching rate of the cell. The statistics turned out to be exactly Gaussian at an arbitrary time of the cell evolution. The Gaussian fluctuations around the average value of the stretching rate vanishes at time tends to infinity. Finally, we described the statistics of the passive scalar field (44) $(\ln(L/r))^{-1} \ll 1)$ with logarithmic accuracy.

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References

[1] Batchelor G K 1959 *J. Fluid Mech.* **5** 113
[2] Kraichnan R 1967 *Phys. Fluids* **10** 1417; 1971 *J. Fluid Mech.* **47** 525; 1975 *J. Fluid Mech.* **67** 155
[3] Falkovich G and Lebedev V 1994 *Phys. Rev.* E **49** R1800
[4] L'vov V 1991 *Phys. Rep.* **207** 1
[5] Kolokolov I 1986 *Phys. Lett.* **114A** 99
[6] Kolokolov I 1990 *Ann. Phys., Lpz.* **202** 165
Kolokolov I and Podivilov E 1989 *JETP* **68** 119
[7] Kolokolov I 1993 *JETP* **76** 1099
[8] Feller W 1957 *An Introduction to Probability Theory and its Applications* (New York: Wiley)
[9] Le Page E 1982 *Probability Measures on Groups* ed A Dold and B Eckmann (Berlin: Springer) p 258
[10] Mel'nikov V I 1980 *JETP Lett.* **32** 225
[11] Fyodorov Y V and Mirlin A D 1993 *JETP Lett.* **58** 615